1.6 – More on Linear Systems and Invertible Matrices

Theorem 1.6.1 A system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

Theorem 1.6.2 If A is an invertible $n \times n$ matrix, then for every $n \times 1$ matrix $\mathbf{b}$, the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

4. Solve the system by inverting the coefficient matrix and using Theorem 1.6.2.

$$\begin{aligned} 5x_1 + 3x_2 + 2x_3 &= 4 \\ 3x_1 + 3x_2 + 2x_3 &= 2 \\ x_2 + x_3 &= 5 \end{aligned}$$

11. Solve the linear systems. Using the given values for the b's solve the systems together by reducing an appropriate augmented matrix to reduced row echelon form.

$$4x_1 - 7x_2 = b_1$$
$$x_1 + 2x_2 = b_2$$

i. $b_1 = 0, b_2 = 1$

ii. $b_1 = -4, b_2 = 6$

iii. $b_1 = -1, b_2 = 3$

iv. $b_1 = -5, b_2 = 1$

17. Determine conditions on the b_i's, if any, in order to guarantee that the linear system is consistent.

$$\begin{aligned} x_1 - x_2 + 3x_3 + 2x_4 &= b_1 \\ -2x_1 + x_2 + 5x_3 + x_4 &= b_2 \\ -3x_1 + 2x_2 + 2x_3 - x_4 &= b_3 \\ 4x_1 - 3x_2 + x_3 + 3x_4 &= b_4 \end{aligned}$$

Theorem 1.6.4 Equivalent Statements (extends Theorem 1.5.3)

If A is an $n \times n$ matrix, then the following are equivalent.

a) A is invertible.

b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

c) The reduced row echelon form of A is I_n.

d) A is expressible as a product of elementary matrices.

e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

Theorem 1.6.5 Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.